SUPNORM AND f-HÖLDER ESTIMATES FOR $\bar{\partial}$ ON CONVEX DOMAINS OF GENERAL TYPE IN $\mathbb{C}^2$

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Abstract. In this paper, we study supnorm and modified Hölder estimates for the integral solution of the $\bar{\partial}$-equation on a class of convex domains of general type in $\mathbb{C}^2$ that includes many infinite type examples.

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Contents

1. Introduction

Let Ω be a smooth, bounded domain in $\mathbb{C}^2$ with 0 in the boundary $b\Omega$. Assume that Ω is strictly convex except possibly on a neighborhood U of 0; and in U, Ω has the form

$$\Omega \cap U = \{\rho(z) = F(|z_1|^2) + r(z) < 0\} \tag{1.1}$$

or

$$\Omega \cap U = \{\rho(z) = F(|\operatorname{Re} z_1|^2) + r(z) < 0\}. \tag{1.2}$$

where F is a strictly increasing, convex function such that $F(0) = 0$, $F(t)/t$ is increasing, and r is convex with $\dfrac{\partial r}{\partial z_2} \neq 0$. We remark that Ω may be of finite type or infinite type since we may choose, for example, $F(t) = t^m$ or $F(t) = \exp(-1/t^\alpha)$. The primary goals of this paper are to investigate the supnorm estimate and develop appropriate Hölder estimate for the integral solution of the $\bar\partial$-equation given by Henkin kernel on a domain Ω satisfying (1.1) or (1.2).

Given a bounded, $\bar\partial$-closed $(0,1)$ form ϕ, the supnorm and the Hölder estimates for the solution of the Cauchy-Riemann equation

$$\bar\partial u = \phi$$

on domain Ω is a fundamental question in several complex variables. A positive answer is well-known when Ω is a

- strongly pseudoconvex domain in $\mathbb{C}^n$ (see [He70], [Ke71], [Ra86] ...),
- convex domain of finite type in $\mathbb{C}^n$ (see [DF06], [DFF99], [H02]...),
- real or complex ellipsoid of finite type in $\mathbb{C}^n$ (see [BC84], [F96], [DFW86],...),
- or a pseudoconvex domain of finite type in $\mathbb{C}^2$ (see [FK88],[Ra90],[CNS92]...).

However, when Ω is of infinite type, the only result is by J. E. Fornaess, L. Lee, and Y. Zhang [FLZ11] who prove supnorm estimates in the case $F(t) = \exp(-1/t^\alpha)$ with $\alpha < \dfrac{1}{2}$ and $r(z) = \operatorname{Re} z_2$ for both (1.1) and (1.2). Denote by $A \lesssim B$ for inequality $A \le cB$ with some positive constant c, for simplification. We denote by $L^\infty(\Omega)$ the space of the essentially bounded functions on Ω and by $\|u\|_\infty$ the essential supremun of $u \in L^\infty(\Omega)$ in Ω.

Theorem 1.1 (Fornaess-Lee-Zhang)**.** *Let Ω be a smooth, bounded domain in $\mathbb{C}^2$ with 0 in the boundary $b\Omega$. Assume that Ω is strictly convex except possibly on a neighborhood U of 0; and in U, Ω has the form*

$$\Omega \cap U = \{\rho(z) = \operatorname{Re} z_2 + \exp(-1/|z_1|^\alpha) < 0\} \tag{1.3}$$

or

$$\Omega \cap U = \{\rho(z) = \operatorname{Re} z_2 + \exp(-1/|\operatorname{Re} z_1|^\alpha) < 0\}. \tag{1.4}$$

with $\alpha < 1$. Then there is a solution to the $\bar\partial$-equation $\bar\partial u = \phi$ for any $\phi \in C^1_{(0,1)}(\bar\Omega)$ and $\bar\partial\phi = 0$, that satisfies $\|u\|_\infty \lesssim \|\phi\|_\infty$.

The first goal of the paper is to prove supnorm estimates on domains satisfying (1.1) or (1.2) which both generalize the class of domains considered in [FLZ11].

Theorem 1.2. *(i) Let Ω and F be as in (1.1). Assume that $\int_0^\delta |\ln F(t^2)|dt < \infty$ for some $\delta > 0$. Then for any bounded $\bar\partial$-closed $(0,1)$ form ϕ on $\bar\Omega$, there is a u such that $\bar\partial u = \phi$ on Ω and $\|u\|_\infty \lesssim \|\phi\|_\infty$.*

(ii) Let Ω and F be as in (1.2). *Assume that* $\int_0^\delta |(\ln t)(\ln F(t^2))|dt < \infty$ *for some* $\delta > 0$. *Then for any bounded* $\bar\partial$*-closed* $(0,1)$ *form* ϕ *on* $\bar\Omega$, *there is a* u *such that* $\bar\partial u = \phi$ *on* Ω *and* $\|u\|_\infty \lesssim \|\phi\|_\infty$.

When Ω is finite type (e.g., $F(t) = t^m$), we known that fractional Hölder estimates hold for both case (1.1) and (1.2). However, when Ω is infinite type (e.g., $F(t) = \exp(-1/t^\alpha)$), McNeal [Mc91] proves the fractional Hölder estimates do not hold.

In this paper, we find a suitable Hölder estimate for infinite type. Let f be an increasing function such that $\lim_{t\to+\infty} f(t) = +\infty$. For $\Omega \subset \mathbb{C}^n$, we define f-Hölder space on Ω by

$$\Lambda^f(\Omega) = \{u : \|u\|_\infty + \sup_{z,z+h\in\Omega} f(|h|^{-1}) \cdot |u(z+h) - u(z)| < \infty\}$$

and set

$$\|u\|_f = \|u\|_\infty + \sup_{z,z+h\in\Omega} f(|h|^{-1}) \cdot |u(z+h) - u(z)|.$$

Note that the f-Hölder spaces include the standard Hölder spaces $\Lambda_\alpha(U)$ by taking $f(t) = t^\alpha$ (so $f(|h|^{-1}) = |h|^{-\alpha}$) with $0 < \alpha < 1$. In this way, f-Hölder spaces generalize the notion of the Hölder spaces.

Since F is strictly increasing F is invertible with inverse F^*. Our main result is

Theorem 1.3. *(i) Let Ω and F be defined by* (1.1). *Assume that* $\int_0^\delta |\ln F(t^2)|dt < \infty$ *for some* $\delta > 0$. *Then for every bounded* $\bar\partial$*-closed* $(0,1)$ *form* ϕ *on* $\bar\Omega$, *there exists a function* u *in* $\Lambda^f(\Omega)$ *such that* $\bar\partial u = \phi$ *and*

$$\|u\|_f \lesssim \|\phi\|_\infty \tag{1.5}$$

where

$$f(d^{-1}) = \Big(\int_0^d \frac{\sqrt{F^*(t)}}{t} dt\Big)^{-1}.$$

(ii) Let Ω and F be defined by (1.2). *Assume that* $\int_0^\delta |(\ln t)(\ln F(t^2))|dt < \infty$ *for some* $\delta > 0$. *Then for every bounded* $\bar\partial$*-closed* $(0,1)$ *form* ϕ *on* $\bar\Omega$, *there exists a function* u *in* $\Lambda^f(\Omega)$ *such that* $\bar\partial u = \phi$ *and*

$$\|u\|_f \lesssim \|\phi\|_\infty \tag{1.6}$$

where

$$f(d^{-1}) = \Big(\int_0^d \frac{\sqrt{F^*(t)}\big|\ln\sqrt{F^*(t)}\big|}{t} dt\Big)^{-1}.$$

The following examples are explicit function f in the choice of F.

Example 1.1. Let $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : F(|z_1|^2) + |z_2 - 1|^2 < 1\}$. Then supnorm and f-Hölder estimates hold for the integral solution of $\bar\partial$-equation in the following examples:

(1) If $F(t^2) = t^{2m}$, then $f(d^{-1}) = d^{-1/2m}$.
(2) If $F(t^2) = 2\exp\left(-\frac{1}{t^\alpha}\right)$ with $0 < \alpha < 1$, then $f(d^{-1}) = (-\ln d)^{\frac{1}{\alpha}-1}$.
(3) If $F(t^2) = 2\exp\left(-\frac{1}{t|\ln t|^\alpha}\right)$ with $\alpha > 1$, then $f(d^{-1}) = \big(\ln(-\ln d)\big)^{\alpha-1}$

Example 1.2. Let $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : F(|\operatorname{Re} z_1|^2) + G(|\operatorname{Im} z_1|^2) + |z_2 - 1|^2 < 1\}$ where $G(t) \equiv 0$ in a neighborhood of 0 and there is a positive constant c such that $t \geq c$ if $G(t) \geq 1$. Then supnorm and f-Hölder estimates hold for the integral solution of the $\bar\partial$-equation in the following examples:

(1) If $F(t^2) = t^{2m}$, then $f(d^{-1}) = d^{-1/2m}|\ln d|^{-1}$.
(2) If $F(t^2) = 2\exp\left(-\frac{1}{t^\alpha}\right)$ with $0 < \alpha < 1$, then $f(d^{-1}) = (-\ln d)^{\frac{1}{\alpha}-1}\big(\ln(-\ln d)\big)^{-1}$.
(3) If $F(t^2) = 2\exp\left(-\frac{1}{t|\ln t|^\alpha}\right)$ with $\alpha > 2$, then $f(d^{-1}) = \big(\ln(-\ln d)\big)^{\alpha-2}$.

Remark 1.4. We remark that superlogarithmic estimates defined by Kohn in [Ko02] for the $\bar\partial$-Laplacian $\Box$ or Kohn-Laplacian $\Box_b$, which imply local hypoellipcity of $\Box$ and $\Box_b$ respectively, hold in both domains defined by (1.1) and (1.2) under hypothesis in Theorem 1.3 (see Appendix).

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2. Preliminaries

In this section we briefly recall the contruction of an integral kernel to solve the $\bar\partial$-equation on convex domains in $\mathbb{C}^2$. Details can be found in [He70], [Ra86].

Let ρ be the defining function of Ω. We can assume that there is a $\delta > 0$ such that $b\Omega \setminus B(0, \delta)$ is strictly convex and

$$\Omega \cap B(0, \delta) = \{\rho(z) = P(z_1) + r(z) < 0\} \tag{2.1}$$

where $P(z_1) = F(|z_1|^2)$ or $F(|\operatorname{Re} z_1|^2)$ and $r(z)$ is convex with $\dfrac{\partial r}{\partial z_2} \neq 0$ on $\Omega \cap B(0, \delta)$.

Define

$$\Phi(z, \zeta) = 2\Big(\frac{\partial\rho(\zeta)}{\partial\zeta_1}(\zeta_1 - z_1) + \frac{\partial\rho(\zeta)}{\partial\zeta_2}(\zeta_2 - z_2)\Big).$$

The following result is a well-known consequence of Taylor's theorem and convexity

Lemma 2.1. *There are suitably small ϵ and c such that*

$$\operatorname{Re}\Phi(z,\zeta) \geq -\rho(z) + \begin{cases} c|z-\zeta|^2 & \zeta \in b\Omega \setminus B(0,\delta) \\ P(z_1) - P(\zeta_1) - 2\operatorname{Re}\dfrac{\partial P}{\partial \zeta_1}(\zeta_1)(z_1-\zeta_1) & \zeta \in b\Omega \cap B(0,\delta) \end{cases} \tag{2.2}$$

for all $z \in \bar{\Omega}$ with $|z-\zeta| \leq \epsilon$.

Choose $\chi \in C^\infty(\mathbb{C}^2 \times \mathbb{C}^2)$ such that $0 \leq \chi \leq 1$, $\chi(z,\zeta) = 1$ for $|z-\zeta| \leq \frac{1}{2}\epsilon$ and $\chi(z,\zeta) = 0$ for $|z-\zeta| \geq \epsilon$. For $j = 1, 2$, define

$$\Phi_j^{\#}(z,\zeta) = \chi \frac{\partial \rho}{\partial \zeta_j}(\zeta) + (1-\chi)(\bar{\zeta}_j - \bar{z}_j).$$

Then

$$\Phi^{\#}(z,\zeta) = \Phi_1(z,\zeta)(\zeta_1 - z_1) + \Phi_2(z,\zeta)(\zeta_2 - z_2)$$

has the following properties for any $\zeta \in b\Omega$:

(i)

$$\operatorname{Re}\Phi^{\#}(z,\zeta) \geq -\rho(z) + \begin{cases} c|z-\zeta|^2 & \zeta \in b\Omega \setminus B(0,\delta), \\ P(z_1) - P(\zeta_1) - 2\operatorname{Re}\dfrac{\partial P}{\partial \zeta_1}(\zeta_1)(z_1-\zeta_1) & \zeta \in b\Omega \cap B(0,\delta); \end{cases} \tag{2.3}$$

for $|z-\zeta| \leq \frac{1}{2}\epsilon$ and $z \in \bar{\Omega}$.

(ii) $\Phi^{\#}(\cdot,\zeta)$ and $\Phi_j(\cdot,\zeta)$, $j = 1, 2$, are holomorphic on $\{z : |z-\zeta| \leq \frac{1}{2}\epsilon\}$.

We now are ready for the integral solution of the $\bar{\partial}$-equation. Let $\phi = \phi_1 d\bar{z}_1 + \phi_2 d\bar{z}_2$ be a bounded $\bar{\partial}$-closed $(0,1)$-form on $\bar{\Omega}$. The Hekin integral solution u of the $\bar{\partial}$-equation $\bar{\partial}u = \phi$ given by

$$u = T\phi(z) = H\phi(z) + K\phi(z).$$

where

$$\begin{aligned} H\phi(z) &= -\frac{1}{2\pi i}\int_{\zeta \in b\Omega} \frac{\Phi_1^{\#}(z,\zeta)(\bar{\zeta}_2 - \bar{z}_2) - \Phi_2^{\#}(z,\zeta)(\bar{\zeta}_1 - \bar{z}_1)}{\Phi^{\#}(z,\zeta)|\zeta - z|^2}\phi \wedge \omega(\zeta); \\ K\phi(z) &= -\frac{1}{2\pi i}\int_{\Omega} \frac{\phi_1(\bar{\zeta}_2 - \bar{z}_2) - \phi_2(\bar{\zeta}_1 - \bar{z}_1)}{|\zeta - z|^4}\omega(\bar{\zeta}) \wedge \omega(\zeta) \end{aligned} \tag{2.4}$$

where $\omega(\zeta) = d\zeta_1 \wedge d\zeta_2$.

It is well known that

$$\|K\phi\|_\infty \lesssim \|\phi\|_\infty \quad \text{and} \quad \|K\phi\|_f \lesssim \|\phi\|_\infty,$$

for any f with $0 < f(d^{-1}) < d^{-1}$ (see Lemma 1.15, page 157 in [Ra86]). Moreover, we have

$$H\phi(z) = -\frac{1}{2\pi i}\int_{\zeta\in b\Omega}\cdots = -\frac{1}{2\pi i}\Big(\int_{\zeta\in b\Omega,|z-\zeta|\le\epsilon}\cdots + \int_{\zeta\in b\Omega,|z-\zeta|\ge\epsilon}\cdots\Big) = -\frac{1}{2\pi i}\int_{\zeta\in b\Omega,|z-\zeta|\le\epsilon}\cdots$$

Since $\Phi_1^{\#}(z,\zeta)(\bar\zeta_2-\bar z_2)-\Phi_2^{\#}(z,\zeta)(\bar\zeta_1-\bar z_1)\equiv 0$ if $|z-\zeta|\ge\epsilon$.
Therefore, it is sufficient to estimate

$$H\phi(z) = -\frac{1}{2\pi i}\int_{\zeta\in b\Omega,|z-\zeta|\le\epsilon}\frac{\Phi_1^{\#}(z,\zeta)(\bar\zeta_2-\bar z_2)-\Phi_2^{\#}(z,\zeta)(\bar\zeta_1-\bar z_1)}{\Phi^{\#}(z,\zeta)|\zeta-z|^2}\phi\wedge\omega(\zeta).$$

We will use the general Hardy-Littewood lemma (see Section 5 below) to obtain the f-Hölder estimates. To do that we need to control the gradient of $T\phi(z)$. We have

$$|\nabla H\phi(z)| \lesssim \|\phi\|_\infty\int_{\zeta\in b\Omega,|z-\zeta|\le\epsilon}\Big(\frac{1}{|\Phi|\cdot|\zeta-z|^2}+\frac{1}{|\Phi|^2\cdot|\zeta-z|}\Big)dS$$

where dS is surface area measure on $b\Omega$. We now use Lemma 2.1 to obtain

$$\int_{\zeta\in b\Omega\setminus B(0,\delta),|z-\zeta|\le\epsilon}\Big(\frac{1}{|\Phi|\cdot|\zeta-z|^2}+\frac{1}{|\Phi|^2\cdot|\zeta-z|}\Big)dS \lesssim \delta^{-1/2}(z).$$

Hence, it remains to estimate

$$L(z) = \int_{\zeta\in b\Omega\cap B(0,\delta),|z-\zeta|\le\epsilon}\Big(\frac{1}{|\Phi|\cdot|\zeta-z|^2}+\frac{1}{|\Phi|^2\cdot|\zeta-z|}\Big)dS.$$

Set $t=\operatorname{Im}\Phi(z,\zeta)$. It is easy to check that $\dfrac{\partial t}{\partial\zeta_2}\neq 0$. So we change coordinate and obtain

$$\begin{aligned}
L(z) \lesssim & \int_{|t|\le\delta,|\zeta_1|<\delta,|z_1-\zeta_1|\le\epsilon}\frac{dt\,d(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)}{(|t|+|\operatorname{Re}\Phi|)(|\rho(z)|^2+|\zeta_1-z_1|^2)}\\
& + \int_{|t|\le\delta,|\zeta_1|<\delta,|z_1-\zeta_1|\le\epsilon}\frac{dt\,d(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)}{(t^2+|\operatorname{Re}\Phi|^2)(|\rho(z)|+|\zeta_1-z_1|)}\\
\lesssim & |\ln(|\operatorname{Re}\Phi|)\cdot\ln\rho(z)| + \int_{|\zeta_1|<\delta,|z_1-\zeta_1|\le\epsilon}\frac{d(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)}{|\operatorname{Re}\Phi|\cdot|\zeta_1-z_1|}\\
\lesssim & |\ln\rho(z)|^2 + \int_{|\zeta_1|<\delta,|z_1-\zeta_1|\le\epsilon}\frac{d(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)}{|\operatorname{Re}\Phi|\cdot|\zeta_1-z_1|}.
\end{aligned} \tag{2.5}$$

Here the last inequality follows by $|\operatorname{Re}\Phi|\ge|\rho(z)|$ for all $\zeta\in b\Omega\cap B(0,\delta)$ and $|z-\zeta|\le\epsilon$ which is itself a consequence of Lemma 2.1 and the convexity of P (see (3.2) and (4.1) below).

We have therefore show

$$|\nabla H\phi(z)| \lesssim \Big(\rho(z)^{-1/2} + \int_{|\zeta_1|<\delta, |\zeta_1 - z_1|<\epsilon} \frac{d(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)}{|\operatorname{Re}\Phi| \cdot |\zeta_1 - z_1|}\Big)\|\phi\|_\infty. \tag{2.6}$$

A similar argument also shows

$$|H\phi(z)| \lesssim \Big(1 + \int_{|\zeta_1|<\delta, |\zeta_1 - z_1|<\epsilon} \frac{\big|\ln|\operatorname{Re}\ \Phi|\big| d(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)}{|\zeta_1 - z_1|}\Big)\|\phi\|_\infty. \tag{2.7}$$

3. Estimates on $\Omega \cap U = \{\rho(z) = F(|z_1|^2) + r(z) < 0\}$

In this section, we give the proof of Theorem 1.2.(i) and Theorem 1.3.(ii). It is sufficient to estimate the integrals in (2.6) and (2.7) when $z \in B(0,\delta)$ so the defining function ρ is of the form $\rho(z) = F(|z_1|^2) + r(z)$ in $B(0,\delta)$.

Lemma 3.1. *Let F be a convex, C^2-smooth function on $[0,\delta]$. Then we have*

$$F(p) - F(q) - F'(q)(p-q) \geq 0$$

for any $p, q \in [0,\delta]$.

The proof is simple and is omitted here.

Lemma 3.2. *For $\delta > 0$ small enough, let F be an invertible on $[0,\delta]$ such that $\frac{F(t)}{t}$ is increasing $[0,\delta]$. Then*

$$\int_0^\delta \frac{dr}{\varrho + F(r^2)} \lesssim \frac{\sqrt{F^*(\varrho)}}{\varrho}$$

for any sufficiently small $\varrho > 0$.

Proof. We split our integration to be two terms

$$\int_0^\delta \frac{dr}{\varrho + F(r^2)} = \int_0^{\sqrt{F^*(\varrho)}} \cdots + \int_{\sqrt{F^*(\varrho)}}^\delta \cdots$$

For the first term, it is easy to see that

$$\int_0^{\sqrt{F^*(\varrho)}} \frac{dr}{\varrho + F(r^2)} \leq \frac{\sqrt{F^*(\varrho)}}{\varrho}.$$

Since $\frac{F(t)}{t}$ is increasing, we have

$$\frac{F(r^2)}{r^2} \geq \frac{F(F^*(\varrho))}{F^*(\varrho)} = \frac{\varrho}{F^*(\varrho)}, \quad \text{or} \quad \frac{F(r^2)}{\varrho} \geq \frac{r^2}{F^*(\varrho)},$$

for any $r \geq \sqrt{F^*(\varrho)}$. Apply this inequality to the second term, we obtain

$$\begin{aligned}\int_{\sqrt{F^*(\varrho)}}^{\delta}\frac{dr}{\varrho+F(r^2)}\leq&\frac{1}{\varrho}\int_{\sqrt{F^*(\varrho)}}^{\delta}\frac{dr}{1+r^2/F^*(\varrho)}\\ \leq&\frac{\sqrt{F^*(\varrho)}}{\varrho}\int_1^{\infty}\frac{dy}{1+y^2}=\frac{\pi}{4}\frac{\sqrt{F^*(\varrho)}}{\varrho}\end{aligned}\tag{3.1}$$

This is complete the proof of Lemma 3.2.

□

Proof of Theorem 1.2.(i). We omit the proof of Theorem 1.2.(i) since it follows in exactly method of the proof of Theorem 1.3.(i) with simpler calculation.

□

Proof of Theorem 1.3.(i). We apply the identity $2\,\mathrm{Re}\,a\bar{b} = |a+b|^2 - |a|^2 - |b|^2$ in (2.2) to obtain

$$\begin{aligned}\mathrm{Re}\,\Phi(z,\zeta)\geq&-\rho(z)+F(|z_1|^2)-F(|\zeta_1|^2)+2F'(|\zeta_1|^2)\,\mathrm{Re}\left(\bar{\zeta}_1(z_1-\zeta_1)\right).\\ \geq&-\rho(z)+\left(F'(|\zeta_1|^2)|z_1-\zeta_1|^2+F(|z_1|^2)-F(|\zeta_1|^2)-F'(|\zeta_1|^2)(|z_1|^2-|\zeta_1|^2)\right)\\ \geq&-\rho(z)+\left(F'(|\zeta_1|^2)|z_1-\zeta_1|^2)\right)\end{aligned}\tag{3.2}$$

where the last inequality follows by Lemma 3.1.

Let $M(z)$ be the integral term in (2.6). We will show that

$$M(z)\lesssim\frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|}\tag{3.3}$$

for $z \in \Omega$. For convenient, set $\varrho = |\rho(z)| > 0$ when $z \in \Omega$. From (3.2), we have

$$M(z)\leq\int_{|\zeta_1|<\delta,|\zeta_1-z_1|<\epsilon}\frac{d(\mathrm{Re}\,\zeta_1)d(\mathrm{Im}\,\zeta_1)}{(\varrho+F'(|\zeta_1|^2)|z_1-\zeta_1|^2)|\zeta_1-z_1|}.$$

There are now two cases.
<u>*Case 1:*</u> $|z_1-\zeta_1| \geq |\zeta_1|$. In this case,

$$(\varrho+F'(|\zeta_1|^2)|z_1-\zeta_1|^2)|z_1-\zeta_1|\geq(\varrho+F'(|\zeta_1|^2)|\zeta_1|^2)|\zeta_1|\geq(\varrho+F(|\zeta_1|^2)|\zeta_1|.$$

Here the last inequality follows from the inequality $tF'(t) \geq F(t)$ which is itself a consequence of the fact that $\dfrac{F(t)}{t}$ is increasing. Therefore, using polar coordinates and Lemma

3.2,

$$\begin{aligned} M(z) &\le \int_{|\zeta_1|<\delta} \frac{d(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)}{(\varrho + F(|\zeta_1|^2))|\zeta_1|} \\ &\lesssim \int_0^\delta \frac{dr}{\varrho + F(r^2)} \\ &\lesssim \frac{\sqrt{F^*(\varrho)}}{\varrho}. \end{aligned} \tag{3.4}$$

<u>*Case 2:*</u> If $|\zeta_1| \ge |z_1 - \zeta_1|$, then the fact that F' is increasing (F is convex) implies

$$F'(|\zeta_1|^2)|z_1 - \zeta_1|^2 \ge F'(|z_1 - \zeta_1|^2)|z_1 - \zeta_1|^2 \gtrsim F(|z_1 - \zeta_1|^2).$$

Similarly, we obtain

$$\begin{aligned} M(z) &\le \int_{|\zeta_1 - z_1|<\epsilon} \frac{d(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)}{(\varrho + F(|\zeta_1 - z_1|^2))|\zeta_1 - z_1|} \\ &\lesssim \int_0^\epsilon \frac{dr}{\varrho + F(r^2)} \\ &\lesssim \frac{\sqrt{F^*(\varrho)}}{\varrho} \end{aligned} \tag{3.5}$$

The proof of (3.3) is complete. Combining (2.6) and (3.3), we obtain

$$|\nabla T(\phi)| \lesssim \frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|}\|\phi\|_\infty \lesssim \frac{\sqrt{F^*(\delta_{b\Omega}(z))}}{\delta_{b\Omega}(z)}\|\phi\|_\infty. \tag{3.6}$$

since the distance $\delta_{b\Omega}(z)$ is comparable to $|\rho(z)|$.

Finally, to apply the general Hardy-Littlewood Lemma (see Section 5), we need to check that $G(t) := \sqrt{F^*(t)}$ satisfies the hypothesis of Theorem 5.1. It is easy to see that $\sqrt{F^*(t)}$ is increasing and $\dfrac{\sqrt{F^*(t)}}{t}$ is decreasing. For δ small enough, $|\ln(F(t^2))|$ is decreasing when $0 \le t \le \delta$ so we can estimate

$$\big|\ln F(\eta^2)\big|\eta \le \int_0^\eta \big|\ln F(t^2)\big|dt \le \int_0^\delta \big|\ln F(t^2)\big|dt < \infty$$

for any $0 \le \eta \le \delta$. The integral is finite by the hypothesis. Consequently, $\sqrt{F^*(t)}|\ln t| < \infty$ for any $0 \le t \le \sqrt{F^*(\delta)}$ and $\lim\limits_{t\to 0} t|\ln F(t^2)| = 0$. This implies

$$\begin{aligned}\int_0^d \frac{\sqrt{F^*(t)}}{t}dt \overset{y:=\sqrt{F^*(t)}}{=} \int_0^{\sqrt{F^*(d)}} y\big(\ln F(y^2)\big)'dy \\ =\sqrt{F^*(d)}\ln d - \int_0^{\sqrt{F^*(d)}}\big(\ln F(y^2)\big)dy < \infty.\end{aligned}\tag{3.7}$$

for d sufficiently small. Here, the integral in (3.7) is finite by the hypothesis. Thus the proof of Theorem 1.3.(i) is complete.

$\square$

4. Estimates on $\Omega \cap U = \{\rho(z) = F(|\mathrm{Re} z_1|^2) + r(z) < 0\}$

In this section, we give the proof of Theorem 1.2.(ii) and Theorem 1.3.(ii). Is is sufficient to estimate the integrations in (2.6) and (2.7) when the defining function of Ω in a neigborhood of 0 has the form $\rho = F(|\operatorname{Re} z_1|^2) + r(z)$.

We set $x_1 = \mathrm{Re}\ z_1$, $y_1 = \mathrm{Im}\, z_1$, $\xi_1 = \mathrm{Re}\ \zeta_1$ and $\eta_1 = \mathrm{Im}\,\zeta_1$. From (2.2), we have

$$\begin{aligned}\mathrm{Re}\ \Phi(z,\zeta) \ge& -\rho(z) + \Big(F(x_1^2) - F(\xi_1^2) - 2F'(\xi_1^2)\xi_1(x_1-\xi_1)\Big)\\ \ge& -\rho(z) + F'(\xi_1^2)(x_1-\xi_1)^2 + \Big(F(x_1^2) - F(\xi_1^2) - F'(\xi_1^2)(x_1^2-\xi_1^2)\Big)\\ \ge& -\rho(z) + F'(\xi_1^2)(x_1-\xi_1)^2\end{aligned}\tag{4.1}$$

where the last inequality follows by Lemma 3.1.

Proof of Theorem 1.2.(ii). We only need to show that the integral term in (2.7) is bounded. By the estimates of Re $\Phi(z,\zeta)$ as above, we get

$$
\begin{aligned}
&\int_{|\zeta_1|<\delta,|\zeta_1-z_1|<\epsilon} \frac{\big|\ln|\mathrm{Re}\ \Phi|\big| d(\mathrm{Re}\,\zeta_1)d(\mathrm{Im}\,\zeta_1)}{|\zeta_1-z_1|} \\
&\lesssim \int_{|\zeta_1|<\delta,|\zeta_1-z_1|<\epsilon} \frac{\big|\ln(F'(\xi_1^2)(x_1-\xi_1)^2)\big| d\xi_1 d\eta_1}{|x_1-\xi_1|+|y_1-\eta_1|} \\
&\lesssim \int_{|x_1|<\delta,|x_1-\xi_1|<\epsilon} \big|\ln|x_1-\xi_1|\cdot\ln(F'(\xi_1^2)(x_1-\xi_1)^2)\big|\, dx_1 \\
&\lesssim \int_{|x_1|<\delta,|x_1-\xi_1|<\epsilon;|\xi_1|\le|x_1-\xi_1|} \cdots + \int_{|x_1|<\delta,|x_1-\xi_1|<\epsilon;|\xi_1|\ge|x_1-\xi_1|} \cdots \qquad (4.2)\\
&\lesssim \int_{|x_1|<\delta} \big|\ln|\xi_1|\cdot\ln(F'(\xi_1^2)\xi_1^2)\big|\, dx_1 \\
&\quad + \int_{|x_1-\xi_1|<\epsilon} \big|\ln|x_1-\xi_1|\cdot\ln(F'((x_1-\xi_1)^2)(x_1-\xi_1)^2)\big|\, dx_1 \\
&\lesssim \int_{|t|<\max\{\delta,\epsilon\}} \big|\ln|t|\cdot\ln(F(t^2))\big|\, dt < \infty
\end{aligned}
$$

Here, the first inequality in the last line of (4.2) follows from the inequality $t^2F(t^2)\ge F(t^2)$ and the last one of this line follows by hypothesis of theorem. This completes the proof of Theorem 1.2.(ii).

□

Proof of Theorem 1.3.(ii). We only need to estimate of the integral term in (2.6). By the estimates of Re $\Phi(z,\zeta)$ above, we observe

$$
\begin{aligned}
&\int_{|\zeta_1|<\delta,|\zeta_1-z_1|<\epsilon} \frac{d\xi_1 d\eta_1}{(\varrho+F'(\xi_1^2)(x_1-\xi_1)^2)(|x_1-\xi_1|+|y_1-\eta_1|)} \\
&\qquad \lesssim \int_{|\xi_1|<\delta,|x_1-\xi_1|<\epsilon} \frac{\big|\ln|x_1-\xi_1|\big|d\xi_1}{\varrho+F'(\xi_1^2)(x_1-\xi_1)^2} \qquad (4.3)\\
&\qquad \lesssim \int_{|t|<\max\{\delta,\epsilon\}} \frac{|\ln t|dt}{\varrho+F(t^2)}
\end{aligned}
$$

Here, the last inequality in the last line of (4.3) follows by the comparison of $|\xi_1|$ and $|x_1-\xi_1|$; and the property $t^2F'(t^2)\ge F(t^2)$ as in Theorem 1.2.(ii). To estimate the integral term in the last line of (4.3) we need following lemma.

Lemma 4.1. *For $\delta > 0$ small enough, let F be an invertible on $[0,\delta]$ such that $\frac{F(t)}{t}$ is increasing $[0,\delta]$. Then*
$$\int_0^\delta \frac{|\ln t|dt}{\varrho + F(t^2)} \lesssim \frac{\sqrt{F^*(\varrho)}|\ln\sqrt{F^*(\varrho)}|}{\varrho}$$
for any $\varrho > 0$ sufficiently small.

Proof of Lemma 4.1. Proof. We split our integration into two terms
$$\int_0^\delta \frac{|\ln t|dt}{\varrho + F(t^2)} = \int_0^{\sqrt{F^*(\varrho)}} \cdots + \int_{\sqrt{F^*(\varrho)}}^\delta \cdots \tag{4.4}$$
For the first term, we have
$$\int_0^{\sqrt{F^*(\varrho)}} \cdots \le \frac{1}{\varrho}\int_0^{\sqrt{F^*(\varrho)}} |\ln t|dt \lesssim \frac{\sqrt{F^*(\varrho)}|\ln\sqrt{F^*(\varrho)}|}{\varrho}.$$
For the second term
$$\int_{\sqrt{F^*(\varrho)}}^\delta \cdots \le |\ln\sqrt{F^*(\varrho)}| \int_{\sqrt{F^*(\varrho)}}^\delta \frac{dt}{\varrho + F(t^2)} \lesssim \frac{\sqrt{F^*(\varrho)}|\ln\sqrt{F^*(\varrho)}|}{\varrho}$$
where the last inequality follows by (3.1). This is the proof of Lemma 4.1.

□

Similarly to the proof of (3.7) we obtain $\int_0^d \frac{\sqrt{F^*(t)}\left|\ln\sqrt{F^*(t)}\right|}{t}dt < \infty$ under hypothesis $\int_0^\delta \left|\ln t \cdot \ln F(t^2)\right|dt < \infty$ for $d, \delta > 0$ enough small.

Using the general Hardy-Littewood Lemma, we obtain the proof of Theorem 1.3.(ii).

□

5. General Hardy-Littewood Lemma for f-Hölder estimates

We conclude by proving a general Hardy-Littlewood Lemma for f-Hölder estimates.

Theorem 5.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let $\delta_{b\Omega}(x)$ denote the distance function from x to the boundary of Ω. Let $G : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function such that $\frac{G(t)}{t}$ is decreasing and $\int_0^d \frac{G(t)}{t}dt < \infty$ for $d > 0$ small enough. If $u \in C^1(\Omega)$ such that*
$$|\nabla u(x)| \lesssim \frac{G(\delta_{b\Omega}(x))}{\delta_{b\Omega}(x)} \quad \textit{for every } x \in \Omega. \tag{5.1}$$

Then $|u(x)-u(y)| \lesssim f(|x-y|^{-1})^{-1}$, *for* $x, y \in \Omega, x \neq y$ *where* $f(d^{-1}) = \Big(\int_0^d \frac{G(t)}{t}dt\Big)^{-1}$.

Remark 5.2. If $G(t) = t^\alpha$, Theorem 5.1 is the usual Hardy-Littlewood Lemma for domains of finite type. The proof of this theorem in this case can be found in [CS01].

Proof. Since $u \in C^1$ in the interior of Ω, we only need to prove the assertion when z and w are near the boundary. Using a partion of unity, we can assume that u is supported in $U \cap \bar{\Omega}$, where U is a neighborhood of a boundary point $x_o \in b\Omega$. After linear change of coordinates, we may assume $x_o = 0$ and for some $\delta > 0$,

$$U \cap \Omega = \{x = (x', x_N) | x_N > \phi(x'), |x'| < \delta, |x_N| \leq \delta\},$$

where $\phi(0) = 0$ and ϕ is some Lipschitz function with Lipschitz constant M. Let $x = (x, x_N), y = (y', y_N) \in \Omega$ and $d = |x-y|$. For $a \geq 0$, we define the line segment L_a by $\theta(x', x_N + a) + (1-\theta)(y', y_N + a)$, $0 \leq \theta \leq 1$. Using the Lipschitz property of ϕ, we obtain

$$\begin{aligned}\tilde{x}_N + Md =& \theta(x_N + Md) + (1-\theta)(y_N + Md) \\ \geq& Md + \theta\phi(x') + (1-\theta)\phi(y') \\ \geq& Md + \theta(\phi(x') - \phi(\tilde{x}')) + (1-\theta)(\phi(y') - \phi(\tilde{x}')) + \phi(\tilde{x}') \\ \geq& \phi(\tilde{x}').\end{aligned} \tag{5.2}$$

This implies that L_a lies in Ω for any $a \geq Md$. Since $u \in C^1(\Omega)$, the Mean Value Theorem tells us that there must exist some $(\tilde{x}', \tilde{x}_N + 2Md) \in L_{2Md}$ such that

$$|u(x', x_N + 2Md) - u(y', y_N + 2Md)| \leq |\nabla u(\tilde{x}', \tilde{x}_N + 2Md)|d.$$

The distance function $\delta_{b\Omega}(x', x_N)$ is comparable to $x_N - \phi(x')$, i.e., there are positive constants c, C such that

$$c(x_N - \phi(x')) \leq \delta_{b\Omega}(x', x_N) \leq C(x_N - \phi(x')) \text{ for } x \in \Omega \cap U. \tag{5.3}$$

Using hypothesis of G, combining with (5.1) and (5.3), it follows that

$$\begin{aligned}|u(x', x_N + 2Md) - u(y', y_N + 2Md)| \lesssim& \frac{G(\delta_{b\Omega}(\tilde{x}', \tilde{x}_N + 2Md))}{\delta_{b\Omega}(\tilde{x}', \tilde{x}_N + 2Md)}d \\ \lesssim& \frac{G(c(\tilde{x}_N + 2Md - \phi(\tilde{x}')))}{c(\tilde{x}_N + 2Md - \phi(\tilde{x}'))}d \\ \lesssim& \frac{G(cMd)}{cMd}d \\ \lesssim& G(d),\end{aligned} \tag{5.4}$$

where the last inequality follows by considering two case of cM; if $cM < 1$, we use $G(t)$ increasing; otherwise, we use $\dfrac{G(t)}{t}$ decreasing. We also have

$$
\begin{aligned}
|u(x) - u(x', x_N + 2Md)| = &\left|\int_0^d \frac{\partial u(x', x_N + 2Mt)}{\partial t} dt\right| \\
\lesssim &\int_0^d \frac{G(\delta_{b\Omega}(x', x_N + 2Mt))}{\delta_{b\Omega}(x', x_N + 2Mt)} dt \\
\lesssim &\int_0^d \frac{G(t)}{t} dt.
\end{aligned}
\tag{5.5}
$$

Thus for any $x, y \in \Omega$,

$$
\begin{aligned}
|u(x) - u(y)| \leq &|u(x) - u(x', x_N + 2Md)| + |u(y) - u(y', y_N + 2Md)| \\
&+ |u(x', x_N + 2Md) - u(y', y_N + 2Md)| \\
\lesssim &G(d) + \int_0^d \frac{G(t)}{t} dt \lesssim \int_0^d \frac{G(t)}{t} dt.
\end{aligned}
\tag{5.6}
$$

Here, the last inequality follows from

$$
G(d) = \int_0^d \frac{G(d)}{d} dt \leq \int_0^d \frac{G(t)}{t} dt.
$$

This proves the theorem.

□

Appendix

In this part, we give an explanation of Remark 1.4. First we show the following theorem.

Theorem 5.3. *Let Ω and F be defined by (1.1) or (1.2) and let $f(d^{-1}) = (\sqrt{F^*(d)})^{-1}$ (for $d > 0$ small enough). Then f-estimate holds for the $\bar\partial$-Neumann problem, that is,*

$$
\|f(\Lambda)u\|^2 \lesssim \|\bar\partial u\|^2 + \|\bar\partial^* u\|^2, \tag{5.7}
$$

for any $u \in C^\infty_{(0,1)}(\bar\Omega) \cap Dom(\bar\partial^*)$, where $\|\cdot\|$ is the $L^2(\Omega)$ norm, $f(\Lambda)$ is the tangential pseudo-differential operator with symbol $f((1 + |\xi|^2)^{1/2})$ and $\bar\partial^*$ is the L^2-adjoint of $\bar\partial$ with its domain $Dom(\bar\partial^*)$.

Proof. We will only give the proof in the case Ω is defined by (1.2), that is,

$$
\Omega \cap U = \{\rho(z) = F(x_1^2) + r(z) < 0\},
$$

since the other proves similarly. Here, $x_1 = \operatorname{Re} z_1$. It is sufficient to show that there exists a family of absolutely bounded weights $\{\Phi^\delta\}$ defined on $S_\delta \cap U$ satisfying

$$\sum_{i,j=1}^{2} \frac{\partial^2 \Phi^\delta}{\partial z_i \partial \bar{z}_j} u_i \bar{u}_j \gtrsim f(\delta^{-1})^2 |u|^2 \quad \text{on} \quad S_\delta \cap U \tag{5.8}$$

for any $u \in C^\infty_{(0,1)}(\bar{\Omega} \cap U)$, where $S_\delta = \{z \in \Omega : -\delta < \rho(z) < 0\}$ and U is a neighborhood of the origin (see Theorem 1.4 in [KZ10]).

For any $\delta > 0$, we define

$$\Phi^\delta(z) := \exp\left(\frac{\rho(z)}{\delta} + 1\right) - \exp\left(-\frac{x_1^2}{4F^*(\delta)}\right).$$

The weights Φ^δ are absolutely bounded on $S_\delta \cap U$. Computing of the Levi form of Φ^δ shows that

$$\begin{aligned}
\sum_{i,j=1}^{2} \frac{\partial^2 \Phi^\delta(z)}{\partial z_i \partial \bar{z}_j} u_i \bar{u}_j =& \frac{1}{\delta} \left(\sum_{i,j=1}^{2} \frac{\partial^2 \rho(z)}{\partial z_i \partial \bar{z}_j} u_i \bar{u}_j + \frac{1}{\delta} \left| \sum_{j=1}^{2} \frac{\partial \rho(z)}{\partial z_j} u_j \right|^2 \right) \exp\left(\frac{\rho(z)}{\delta} + 1\right) \\
&+ \frac{1}{8F^*(\delta)} \left(1 - \frac{x_1^2}{2F^*(\delta)}\right) \exp\left(-\frac{x_1^2}{4F^*(\delta)}\right) |u_1|^2 \\
\geq& \frac{1}{2} \left[\frac{1}{\delta} \frac{F(x_1^2)}{x_1^2} + \frac{1}{4F^*(\delta)} \left(1 - \frac{x_1^2}{2F^*(\delta)}\right) \exp\left(-\frac{x_1^2}{4F^*(\delta)}\right) - \frac{c}{F^*(\delta)} \right] |u_1|^2 \\
&+ cF^*(\delta)^{-1} |u_2|^2,
\end{aligned} \tag{5.9}$$

for any $z \in S_\delta \cap U$, where $c > 0$ will be chosen small. Here, the inequality follows by the hypothesis of ρ and F.

We use the notation that

$$A = \frac{1}{\delta} \frac{F(x_1^2)}{x_1^2}, \quad B = \frac{1}{4F^*(\delta)} \left(1 - \frac{x_1^2}{2F^*(\delta)}\right) \exp\left(-\frac{x_1^2}{4F^*(\delta)}\right), \quad \text{and } C = -\frac{c}{F^*(\delta)}.$$

We consider two cases.

Case 1. $x_1^2 \leq F^*(\delta)$. We have $B \geq \dfrac{e^{-1/4}}{8F^*(\delta)}$, and hence $A + B + C \gtrsim (F^*(\delta))^{-1}$ for a small choice of c in term C.

Cases 2. Otherwise, assume $x_1^2 \geq F^*(\delta)$. Using our assumption that $\dfrac{F(t)}{t}$ is increasing, we get

$$A = \frac{1}{\delta} \frac{F(x_1^2)}{x_1^2} \geq \frac{1}{\delta} \frac{F(F^*(\delta))}{F^*(\delta)} = \frac{1}{\delta} \frac{\delta}{F^*(\delta)} = \frac{1}{F^*(\delta)}$$

In this case, B can be negative; however, by using the fact that $\min_{t\geq 1/2}\left\{(1-t)e^{-t/2}\right\}=-2e^{-3/2}$ for $t=\dfrac{x_1^2}{2F^*(\delta)}\geq\dfrac{1}{2}$, we have $B\geq-\dfrac{e^{-3/2}}{2F^*(\delta)}$. This implies $A+B+C\gtrsim(F^*(\delta))^{-1}$ for c small enough

Therefore, we obtain (5.8). That concludes the proof of Theorem 5.3.

□

By the equivalence of an f-estimate on a domain and its boundary in $\mathbb{C}^2$ (see [Kh10]), we have

$$\|f(\Lambda)u\|^2\lesssim\|\bar\partial_b u\|^2+\|\bar\partial_b^* u\|^2 \tag{5.10}$$

holds for any $u\in C^\infty_{(0,0)}(b\Omega)\cap\ker(\bar\partial_b)$ or $u\in C^\infty_{(0,1)}(b\Omega)\cap\ker(\bar\partial_b^*)$. Here, the norm in (5.10) is L^2-norm on $b\Omega$ and $\bar\partial_b$ is tangential Cauchy-Riemann operator on $b\Omega$ with its adjoint $\bar\partial_b^*$.

Next, we notice that the hypothesis in Theorem 1.3 implies $\lim\limits_{t\to 0}(t\ln F(t^2))=0$. This limit is equivalent to $\lim\limits_{\delta\to 0}\dfrac{f(\delta^{-1})}{\log\delta}=\infty$. This means superlogarihmic estimates (in the sense of Kohn [Ko02]) for $\Box$ and $\Box_b$ hold. Until now, we did not know if there is a function f such that f-Hölder estimate for the integral solution of the $\bar\partial$-equation on Ω (defined by (1.1) or (1.2)) holds when $F(t^2)=\exp\left(-\dfrac{1}{t|\ln t|^\alpha}\right)$ with $0<\alpha\leq 1$. However, in this case, L^2-superlogarithmic estimates for $\Box$ and $\Box_b$ still hold.

References

[BC84] **J. Bruna, J. Castillo**—Hölder and L^p estimates for the $\bar\partial$ equation in some convex domains with real analytic boundary, *Math. Ann.*, **269** (1984), 527–539.

[CNS92] **D. C. Chang, A. Nagel, E. M. Stein**— Estimates for the $\bar\partial$-Neumann problem in pseudoconvex domains of finite type in $\mathbb{C}^2$, *Acta Math.*, **169**(1992), 153–228.

[CS01] **S. C. Chen, M. C. Shaw**—Partial differential equations in several complex variables, *AMS/IP Studies in Advanced Mathematics, 19. American Mathematical Society, Providence, RI; International Press, Boston, MA,* (2001), xii+380 pp.

[DF06] **K. Diederich, B. Fischer**—Hölder estimates on lineally convex domains of finite type, *Michigan Math. J.*, **54** (2006), 341–352.

[DFF99] **K. Diederich, B. Fischer, J. E. Fornaess**—Hölder estimates on convex domains of finite type, *Math. Z.*, **232** (1999), 43–61.

[DFW86] **K. Diederich, J. E. Fornaess, J. Wiegerinck**— Sharp Hölder estimates for $\bar\partial$ on ellipsoids, *Manuscripta Math.*, **56** (1986), 399–413.

[F96] **J. F. Fleron**— Sharp Hölder estimates for $\bar\partial$ on ellipsoids and their complements via order of contact, *Proc. Amer. Math. Soc.*, **124** (1996), 3193–3202.

[FK88] **C. L. Fefferman, J. J. Kohn**— Hölder estimates on domains of complex dimension two and on three dimensional CR manifolds, *Adv. Math.*, **62** (1988), 223–303.

[FLZ11] **J. E. Fornaess, L. Lee, Y. Zhang**—On suporm estimates for $\bar{\partial}$ on infinite type convex domains in $\mathbb{C}^2$, *J. Geom. Anal.*, **21** (2011), 495–512.

[Ke71] **N. Kerzman**—Hölder and L^p estimates for solutions of $\bar{\partial}u = f$ in strongly pseudoconvex domains, *Comm. Pure Appl. Math.*, **24** (1971), 301–379.

[Kh10] **T. V. Khanh**—A general method of weights in the $\bar{\partial}$-Neumann problem. *Ph.D. thesis. arXiv:1001.5093v1* (2010).

[Ko02] **J. J. Kohn**—Superlogarithmic estimates on pseudoconvex domains and CR manifolds. *Ann. of Math.*, **156** (2002), 213248.

[Kr76] **S. Krantz**—Optimal Lipschitz and L^p estimates for the equation $\bar{\partial}u = f$ on strongly pseudoconvex domains, *Math. Ann.*, **219** (1976) , 233–260.

[KZ10] **T. V. Khanh, G. Zampieri**—Regularity of the $\bar{\partial}$-Neumann problem at point of infinite type. *J. Funct. Anal.*, **259** (2010), 2760–2775.

[H02] **T. Hefer**— Hölder and L^p estimates for $\bar{\partial}$ on convex domains of finite type depending on Catlin's multitype, *Math. Z.*, **242** (2002), 367–398.

[Mc91] **J. D. McNeal**—On sharp Hölder estimates for the solutions of the $\bar{\partial}$-equations. Several complex variables and complex geometry, Part 3 (Santa Cruz, CA, 1989), 277-285, Proc. Sympos. Pure Math., 52, Part 3, Amer. Math. Soc., Providence, RI, 199l.

[He70] **G. M. Henkin**—Integral representation of functions in strictly pseudoconvex domains and applications to the $\bar{\partial}$-problem, *Mat. Sb.*, **124**(1970), 300–308.

[Ra86] **R. M. Range**—Holomorphic Functions and Integral Representations in Several Complex Variables, Springer-Vedag, Berlin/New York, 1986.

[Ra90] **R.M. Range**— Integral kernels and Hölder estimates for $\bar{\partial}$ on psudoconvex domains of finite type in $\mathbb{C}^2$, *Math. Ann.*, **288** (1990), 63–74.